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Symmetry breaking Skyrme models in $\mathbb{R}_d$

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Abstract. We present models involving an anti-Hermitian scalar field Φ, endowed with a symmetry breaking potential. Like the Skyrme model, these are invariant under the global unitary transformation $\Phi \to A\Phi A^{-1}$ and have topologically stable localized solutions. Unlike the Skyrme model, these are defined on every $\mathbb{R}_d$, and a subclass of these models possess minimal action explicit solutions with winding number n.

1. Introduction

The Skyrme model [1] is a nonlinear field theory of pions, which has been employed as an effective field theory to describe QCD at low energy [2].

In the static limit, it has topologically stable finite energy solutions in $\mathbb{R}_3$. These are treated as the solitons of the $(3+1)$-dimensional theory, and in the standard treatment [2] quantization is carried out about this classical background using the method of collective coordinates. It turns out that this skyrmion can be quantized as a fermion. Physically, this is the most important feature of Skyrme theory, and can be ascribed most simply to its invariance, under the global transformation

$$U \to AUA^{-1} \tag{1}$$

of the Skyrme field $U = \exp i\boldsymbol{\pi}\cdot\boldsymbol{\sigma}$.

Invariance under (1) is an essential property of the Skyrme model, and in what follows it will be satisfied by the new models we propose. These models differ from the Skyrme model in two qualitative ways. The first is that they are described by Hermitian (or antihermitian) fields Φ which in the static limit are subject to the asymptotic condition

$$\operatorname{tr}\Phi^2 \xrightarrow[r\to\infty]{} \eta^2 \tag{2}$$

at spatial infinity, by virtue of the presence of a symmetry breaking potential. The second is that unlike the (static) Skyrme model, which is defined only on $\mathbb{R}_3$, these new models form a hierarchy which can be defined on every $\mathbb{R}_n$. This last feature is not physically relevant, but is interesting for its own sake.

Before we give the construction of the new models in section 3, we re-examine the Skyrme model from a view point that is particularly pertinent to its comparison with the former in section 2. After that we construct the models and proceed to present their soliton solutions in section 4.

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2. The Skyrme model

Our approach here is indirect, in that we first construct $O(d+1)$ sigma models in d dimensions, and then note that the $O(4)$ model in three dimensions is the Skyrme model.

The $O(d+1)$ sigma models in d dimensions are defined, following [3], in terms of the order parameter field $n^\alpha(x)$, $\alpha = 1, \ldots, d+1$, depending on the $\mathbb{R}_d$ coordinate x_i, $i = 1, \ldots, d$, and n^α satisfy the constraint

$$n^\alpha n^\alpha = 1. \tag{3}$$

In terms of the totally antisymmetric tensor fields

$$\hat{F}^{\alpha_1 \ldots \alpha_n}_{i_1 \ldots i_n} = \partial_{[i_1} n^{\alpha_1} \ldots \partial_{i_n]} n^{\alpha_n} \tag{4}$$

one can write down the following inequality

$$\int^\alpha \mathrm{d}^d x [\kappa^{(m-n)} \varepsilon_{i_1 \ldots i_m j_1 \ldots j_n} \hat{F}^{\alpha_1 \ldots \alpha_n}_{j_1 \ldots j_n} - \varepsilon_{\alpha_1 \ldots \alpha_n \beta_1 \ldots \beta_{m+1}} n^{\beta_{m+1}} \hat{F}^{\beta_1 \ldots \beta_m}_{i_1 \ldots i_m}]^2 \geqslant 0 \tag{5}$$

where κ is a constant with the dimensions of an inverse length to the $(m-n)$th power, and where $m \geqslant n$ with $m+n=d$.

As a result of (5) we have the topological inequality

$$\int \mathscr{S}_{n,m} \, \mathrm{d}^d x \geqslant \int \frac{2n!\,m!\,\kappa^{(m-n)}}{(n+m)!} \varepsilon_{i_1 \ldots i_d} \varepsilon_{\alpha_1 \ldots \alpha_{d+1}} n^{\alpha_{d+1}} \hat{F}^{\alpha_1 \ldots \alpha_d}_{i_1 \ldots i_d} \tag{6}$$

with

$$\mathscr{S}_{n,m} = m!\,\kappa^{2(m-n)} (\hat{F}^{\alpha_1 \ldots \alpha_n}_{i_1 \ldots i_n})^2 + n!\,(\hat{F}^{\alpha_1 \ldots \alpha_m}_{i_1 \ldots i_m})^2. \tag{7}$$

For field configurations with the appropriate (instantonic) asymptotic properties [3, 4], the right-hand-side of (6) becomes a winding number, which gives a lower bound on the positive-definite action on the left of (6). Thus the Euler–Lagrange equations of the system defined by the density (7) can have non-trivial and topologically stable (instanton) solutions.

These solutions are in general non-minimal, in the sense that they do not saturate the inequality (6). In the special case, however, when $d = 2p$ is even, one can choose $m = n$ as a consequence of which the dimensional constant κ does not feature in (7), and $\mathscr{S}_{n,n}$ in $d(=2n)$ dimensions is conformally invariant. This case was studied in detail in [3], and minimal action (self-dual) instanton solutions were found.

These instanton field configurations [3]

$$n^\alpha = \frac{2ax^g\alpha}{x^2+a^2} \qquad n^{d+1} = \frac{x^2-a^2}{x^2+a^2} \tag{8}$$

depend on an arbitrary scale parameter a, and hence are not localized as solitons. For this reason we exclude these special cases from our consideration.

Excluding $m = n$, therefore, all systems $\mathscr{S}_{n,m}$ given by (7) have topologically stable localized soliton solutions, localized to the scale of the dimensional parameter κ. On this purely classical level, they can be interpreted as Skyrme-like models on $\mathbb{R}_{(m+n)}$.

As was noted above, all these Skyrme-like models have only non-minimal solutions. This can be seen simply by verifying that the first-order (anti-)self-duality equations, that render the inequality (5) into an equality, have no spherically symmetric solutions. This is a consequence of the presence of κ, the scale breaking constant in (7). Indeed the relevant (anti-)self-duality equations can be solved on $\mathscr{S}^{n+m}$, rather than $\mathbb{R}_{n+m}$, since the dimensional constant κ^{-1} can then be identified with the (constant) radius of $\mathcal{S}^{n+m}$ and this disappears from the (anti-)self-duality equation. In the case of the Skyrme model on $\mathcal{S}^3(n=1,\ m=2)$, those aspects have been thoroughly discussed in [5, 6]. It would in fact be expected that this situation will hold on any compact symmetric coset space, and not just spheres. This is completely analogous to finite action solitons of the generalized Yang-Mills (GYM) systems [7], on even-dimensional manifolds. There too only on $\mathbb{R}_{4p}$, when the action is conformally invariant, it is that the (anti-)self-dual solutions exist [8], and not on $\mathbb{R}_{4p+2}$. By contrast (anti-)self-dual solutions exist, and we explicitly found them [9, 10], not only on S^{4p} and $\mathbb{CP}^{2p}$ but also on S^{4p+2} and $\mathbb{CP}^{2p+1}$.

On the quantum level, however, because the $O(n+m+1)$ sigma models are devoid of any invariance under a transformation like (1), they are not interesting as candidates for Skyrme-like models.

The unique exception is the $(n=1,\ m=2)$ case, namely the O(4) sigma model on $\mathbb{R}_3$, which turns out to be (classically) equivalent to the Skyrme model. In this case only, the order parameter field n^α can be used to parametrize the Skyrme field

$$U=n_\alpha\sigma_\alpha \qquad U^{-1}=U^\dagger=n_\alpha\tilde{\sigma}_\alpha \tag{9a, b}$$

with $\sigma_\alpha=(1,\mathrm{i}\sigma_i)$, $\tilde{\sigma}_\alpha=(1,-\mathrm{i}\sigma_i)$ in terms of the Puali matrices σ_i. The three independent components of n^α parametrize the isotriplet pion field π^i. With (9a, b), the (classical) equivalence of the (static) Skyrme Lagrange density

$$\mathcal{L}_0=\kappa^2\,\mathrm{tr}\,u_i^2+\mathrm{tr}\,u_{ij}^2 \tag{10}$$

$$u_i=U^{-1}\partial_i U \qquad u_{ij}=[u_i,u_j] \tag{10a, b}$$

with the density $\mathcal{S}_{1,2}$ of (7), namely that $\mathcal{L}_0\sim\mathcal{S}_{1,2}$, is easy to verify.

We have highlighted the uniqueness of the Skyrme model, or the O(4) sigma model, on $\mathbb{R}_3$, with a view to contrasting this aspect with the situation for the new models to be introduced below. For those, the dimensionality of the space does not occupy a special role.

We complete our discussion of the Skyrme model by pointing out that $\mathcal{S}_{1,2}$ in (10) can be further augmented by a sextic term, as in [6, 11]

$$\mathcal{S}'_{1,2}=\lambda^2\,\mathrm{tr}\,u_{ijk}^2 \tag{11}$$

$$u_{ijk}=\{u_i,u_{jk}\}+\mathrm{cycl}(i,j,k) \tag{11a}$$

with the parameter λ of the dimensions of length. The addition of $\mathcal{S}'_{1,2}$ to $\mathcal{S}_{1,2}$ does not invalidate the topological inequality (6), even though the latter does not follow from (5) any more. In our new models, qualitatively similar terms will feature.

Perhaps the greatest difference between our models and the $O(d+1)$ sigma models will be the expression for the topological charge. Unlike the right-hand side of (6) for the sigma models, these will be expressed as surface integrals, by virtue of the symmetry breaking mechanism.

3. Construction of models

The systematic approach essentially involves the statement of virial theorems employed in the case [12] of YM-Higgs (YMH) and GYMH systems [13] on $\mathbb{R}_d$.

Using the property $\partial_\mu \Theta^{\mu\nu}=0$ of the gauge-invariant stress tensor $\Theta^{\mu\nu}$, one arrives at the identiy [12]

$$\int \Theta^\mu_\mu \, d^d x = 0. \tag{12}$$

The models we will propose below are subsystems of the GYMH systems, obtained by setting the curvature field $F=0$. These can be arrived at without reference to the GYM systems, but the present approach is more instructive. Indeed with a view to incorporating fermions eventually, the GYMH approach is indispensible for the definition of the corresponding index theorems.

Now the GYMH systems in question can be regarded as the residual systems on $\mathbb{R}_d \times K$, with K a coset space, after integrating out the dependence of the action on the coordinates of K, that is, after performing coset-space dimensional reduction [14, 15].

Denoting the components of the curvature $\mathscr{F}_{MN}=(\mathscr{F}_{\mu\nu}, \mathscr{F}_{\mu m}\mathscr{F}_{mn})$ on $\mathbb{R}_d \times K$, with $x_\mu \in \mathbb{R}_d$ and $x_m \in K$, we can summarize the result of the dimensional reduction by noting the dependence of $\mathscr{F}_{MN}$ on x_μ:

$$\mathscr{F}_{\mu\nu} = F_{\mu\nu} \otimes 1 \tag{13a}$$

$$\mathscr{F}_{\mu m} = D_\mu \Phi(x) \otimes \tfrac{1}{2}\Gamma_m \tag{13b}$$

$$\mathscr{F}_{mn} = S \otimes \Gamma_{mn} \qquad S = \eta^2 + \Phi^2. \tag{13c}$$

Here we have ignored the dependence of $\mathscr{F}_{MN}$ on x_m, and $F_{\mu\nu}$ is the curvature of the connection A_μ on $\mathbb{R}_d$, interacting, minimally, with the Higgs field Φ, endowed with a symmetry breaking term coming from (13c). Γ_m and $\Gamma_{mn} = -\frac{1}{4}[\Gamma_m, \Gamma_n]$ are Γ-matrices.

Starting with the GYM system

$$A = \int_{\mathbb{R}_d \times K^{4p-d}} \operatorname{tr} \mathscr{F}(2p)^2 \tag{14}$$

$$\mathscr{F}(2p) = \mathscr{F} \wedge \ldots \wedge \mathscr{F} \qquad p\text{-times} \tag{14a}$$

the dependence of the residual action A_R on (A_μ, Φ) can be symbolically recorded as

$$A_R = \|F^p\|^2 + \|F^{p-1} D\Phi\|^2 + \ldots + \|S^{p-2} D\Phi^2 + S^{p-1} F\|^2 + \|S^{p-1} D\Phi\|^2 + \|S^p\|^2 \tag{15}$$

where non-trivial models can arise only for $p \geqslant 2$. Note that every term in (15) has the same dimension, and that in all products and powers of fields, all possible permutations and antisymmetrization of all vector indices is implied.

Substituting the action density in (15), into the identity (12) we have the virial theorem, similarly expressed

$$(d-4p)\|F^p\|^2 + (d-4p+2)\|F^{p-1} D\Phi\|^2 + (d-4)\|S^{p-2} + D\Phi^2 + S^{p-1} F\| + \ldots$$
$$+ (d-2)\|S^{p-1} D\Phi\|^2 + (d-0)\|S^p\|^2 = 0. \tag{16}$$

For $p=1$, (16) is familiar [12] and states that finite action YMH field configurations, in $d=4$ can only have constant Higgs field $\Phi=\eta$, in $d=3, 2$ *must* have nontrivial Φ and A_μ, and in $d=1$ only Φ (and no curvature.) For $p=2$, (16) was used to study GYMH field configurations in [13, 16].

Here we are interested in models devoid of gauge fields, defined exclusively in terms of the scalar field Φ. We note that, for $F=0$, all terms in (16), down to and excluding the one with coefficient $(d-2p)$, vanish. It is therefore impossible to find finite action field configurations with $F=0$ for $d \geqslant 2p-1$. For example for $p=1, 2, 3$, equation (16) with $F=0$ reads as

$$(d-2)\|\partial\Phi\|^2+d\|S\|^2=0 \tag{17a}$$

$$(d-4)\|\partial\Phi^2\|^2+(d-2)\|S\partial\Phi\|^2+d\|S^2\|^2=0 \tag{17b}$$

$$(d-6)\|\partial\Phi^3\|^2+(d-4)\|S\partial\Phi^2\|^2+(d-2)\|S^2\partial\Phi\|^2+d\|S^3\|^2=0. \tag{17c}$$

Whenever each term in (*17a, b, c*) is positive, finite action field configurations are not viable, that is, when in (a) $d>1$, in (b) $d>3$ and in (c) $d>5$. In these particular examples, the viable models respectively for (a) in $d=1$, for (b) in $d=1,2,3$ and for (c) in $d=1,\ldots,5$ are

$$\mathscr{L}_a=\Phi_i^2+S^2 \tag{18a}$$

$$\mathscr{L}_b=\Phi_{ij}^2+(S\Phi_i+\Phi_iS)^2+S^4 \tag{18b}$$

$$\mathscr{L}_c=\Phi_{ijk}^2+(S\Phi_{ij}+\Phi_{[i}S\Phi_{j]}+\Phi_{ij}S)^2d+(S^2\Phi_i+S\Phi_iS+\Phi_iS^2)+S^6 \tag{18c}$$

with

$$\Phi_i=\partial_i\Phi \tag{19a}$$

$$\Phi_{ij}=[\Phi_i, \Phi_j] \tag{19b}$$

$$\Phi_{ijk}=\{\Phi_i, \Phi_{jk}\}+\text{cycl } ijk. \tag{19c}$$

The hierarchy of models promised can be illustrated, by its first three members, by the Lagrange densities $(18a\text{--}c)$. Furthermore, the precise form of each can be altered subject to (i) that $(18a\text{--}c)$ is bounded from below by a topological charge density, i.e. a total divergence, and (ii) that the potential terms $S^{2n}=(\eta^2+\Phi^2)^{2n}$ be altered only such that the replacement of $V=S^{2n}$ is also a symmetry breaking potential, e.g. $V=(\eta^2+\Phi^2)^{2m}$, $m\neq n$, or $V=(1+\cos\Phi)$, etc, with $\operatorname{tr}\Phi^2\to-\eta^2$ asymptotically, to ensure topological stability in the former case.

Subject to these two restrictions (i) and (ii), $(18a\text{--}c)$ can be arbitrarily altered. To illustrate the important restriction (i), we list the topological inqualities pertaining to each case $(18a\text{--}c)$.

In dimensions $d=1$, 2, it was found in [17] that non-trivial field configurations occur only for scalar valued Φ. Here we are primarily concerned with analogues of the Skyrme model, and hence consider only non-Abelian Φ in $d>2$.

In $d=3$, the inequality

$$\operatorname{tr}[\Phi_{ij}-\tfrac{1}{2}\varepsilon_{ijk}(S\Phi_k+\Phi_kS)]^2\geqslant 0 \tag{20}$$

leads to

$$tr[\Phi_{ij}^2+(S\Phi_k+\Phi_kS)^2]\geqslant\tfrac{1}{3}\varepsilon_{ijk}\operatorname{tr}S\Phi_{ijk} \tag{20$'$}$$

which supplies $\mathscr{L}_b$, with or iwhtout a potential term $V=S^4$, in $(18b)$ with a topological lower bound. Again in $d=3$, the inequalities

$$\operatorname{tr}\left[\Phi_{ijk}-\frac{1}{3!}\varepsilon_{ijk}S^3\right]^2\geqslant 0 \tag{21}$$

$$\operatorname{tr}[(S\Phi_{ij}+\Phi_{[i}S\Phi_{j]}+\Phi_{ij}S)-\tfrac{1}{2}\varepsilon_{ijk}(S^2\Phi_k+S\Phi_kS+\Phi_kS^2)]^2\geqslant 0 \tag{22}$$

lead to

$$\mathrm{tr}[\Phi_{ijk}^2 + S^6] \geq \frac{2}{3!}\,\varepsilon_{ijk}\,\mathrm{tr}\,S^3\Phi_{ijk} \tag{21'}$$

$$\begin{aligned}\mathrm{tr}[&(S\Phi_{ij} + \Phi_{[i}S\Phi_{j]} + \Phi_{ij}S)^2 + (S^2\Phi_k + S\Phi_kS + \Phi_kS^2)] \\ &\geq \varepsilon_{ijk}\,\mathrm{tr}[\tfrac{3}{2}(S\Phi_{ij}S^2\Phi_k + S^2\Phi_{ij}S\Phi_k) + \tfrac{1}{3}S^3\Phi_{ijk} + S\Phi_iS\Phi_jS\Phi_k].\end{aligned} \tag{22'}$$

The left-hand sides of (21′), and (22′) with or without an additional potential term $V = V(S)$, can respectively be identified with the action densities of two distinct new models in $\mathbb{R}_3$.

The inequalities (21′) and (22′) can also be used to express a topological lower bound on the action density $\mathscr{L}_c$ given by (18*c*), when $d = 3$.

In $d = 4$, the inequalities

$$\mathrm{tr}\left[\Phi_{ijk} - \frac{1}{3!}\,\varepsilon_{ijkl}(S^2\Phi_l + S\Phi_lS + \Phi_lS^2)\right]^2 \geq 0 \tag{23}$$

$$\mathrm{tr}\left[(S\Phi_{ij} + \Phi_{[i}S\Phi_{j]} + \Phi_{ij}S) - \frac{1}{2!}\,\varepsilon_{ijkl}(S\Phi_{kl} + \Phi_{[k}S\Phi_{l]} + \Phi_{kl}S^2)\right]^2 \geq 0 \tag{24}$$

lead to

$$\mathrm{tr}[\Phi_{ijk}^2 + (S^2\Phi_i + S\Phi_iS + \Phi_iS^2)^2] \geq 2\varepsilon_{ijkl}\,\mathrm{tr}\,S\Phi_i\Phi_j\Phi S\Phi_l \tag{23'}$$

$$\mathrm{tr}[S\Phi_{ij}\Phi_{[i}S\Phi_{j]} + \Phi_{ij}S]^2 \geq \varepsilon_{ijkl}\,\mathrm{tr}\,S\Phi_i\Phi_j\Phi_k\Phi_l \tag{24'}$$

Again, the left-hand sides of (23′) and (24′) can (with or without an additional potential $V = V(S)$) be identified with two distinct action densities in $\mathbb{R}_4$. Again, (23′) and (24′) supply the topological lower bound on $\mathscr{L}_c$ in (18*c*).

In $d = 5$, the inequality

$$\mathrm{tr}\left[\Phi_{ijk} - \frac{1}{3!}\,\varepsilon_{ijklm}(S\Phi_{lm} + \Phi_{[l}S\Phi_{m]} + \Phi_{lm}S)\right]^2 \geq 0 \tag{25}$$

leads to

$$\mathrm{tr}[\Phi_{ijk}^2 + (S\Phi_{ij} + \Phi_{[i}S\Phi_{j]} + \Phi_{ij}S)^2] \geq 2\varepsilon_{ijklm}\,\mathrm{tr}\,S\Phi_i\Phi_j\Phi_k\Phi_l\Phi_m. \tag{25'}$$

Yet again, the left-hand side of (25′) can be identified with the action density, bounded from below, for a new model on $\mathbb{R}_5$, and (25′) can be used to establish the topological lower bound of $\mathscr{L}_c$ in (18*c*).

It is clear this process can be continued indefinitely, giving rise to models like (18*a*–*c*) defined in all $\mathbb{R}_d$. All these systems are guaranteed to have classical solutions with finite actions, by virtue of the fact that their actions are all bounded from below by topological charges, which are the integrals of densities like those occurring on the right-hand sides of (20′)–(25′). That these latter are nontrivial topological charges follows from the fact that each of the densities (20′)–(25′) are total divergences by virtue of the symmetry-breaking stipulation on S made above, and from our requirement of the asymptotic condition (2).

Like the Skyrmion in $\mathbb{R}_3$, these new field configurations in $\mathbb{R}_d$ are also localized to an absolute scale, in this case η occurring in S.

Fianlly, we note that the 'energy' of these new solitons in $\mathbb{R}_d$ is unchanged under the global transformations

$$\Phi \to A\Phi A^{-1} \tag{26}$$

as in the Skyrmion case (1). This last property will decide the quantum properties of the new models, and in the $d=3$ examples, the new models can apparently play the same rôle [2] as the Skyrme model does in low-energy QCD.

4. Solutions

Field configurations satisfying the Euler-Lagrange equations of our new models, e.g. (18a-c) and their special cases given by (20′)-(25′), are in general non-minimal. They do not satisfy first-order equations saturating the inequalities (20)-(25), except in very special cases. Except in these latter cases, one cannot expect to find explicit solutions. This is not very unexpected, because the Bogomoln'yi equations saturating the inequalities (20)-(25) are matrix-valued equations of various tensor ranks. On the other hand, the only functions subjected to the variational principle are the matrix valued scalar fields Φ of zero tensor rank. It is therefore conceivable that the Bogomoln'yi equations are overdetermined. In any case, we have explicitly verified, in some examples, that this is so and the Bogomoln'yi equations are solved only by the trivial fields $\Phi=0$, except in some very special models to be presented below.

Before proceeding with the special models whose minimal action solutions can be explictly found, we examine a non-minimal model in detail, by way of illustration.

For this purpose, we choose the model whose action density is defined by the left-hand side of (20′) on $\mathbb{R}_3$. The Bogomoln'yi equation is

$$[\partial_i\Phi, \partial_j\Phi] = \tfrac{1}{2}\varepsilon_{ijk}(S\partial_k\Phi + \partial_k\Phi S) \tag{27}$$

and the topological density is

$$\rho = \varepsilon_{ijk}\,\mathrm{tr}(\eta^2+\Phi^2)\partial_i\Phi\partial_j\Phi\partial_k\Phi \tag{28}$$

which, like all the topological densities featured in the right-hand sides of (20′)-(25′) is a total divergence

$$\rho = \mathrm{i}\partial_i\varepsilon_{ijk}\,\mathrm{tr}[-\eta^2\Phi\partial_j\Phi\partial_k\Phi - \tfrac{2}{5}\Phi^3\partial_j\Phi\partial_k\Phi + \tfrac{1}{5}\Phi^2\partial_j\Phi\Phi\partial_k\Phi]. \tag{28′}$$

Let now Φ be a 2×2 Hermitian matrix, whence the most general radial ansatz is

$$\Phi = k(r) + f(r)\hat{x}\cdot\boldsymbol{\sigma}. \tag{29}$$

For Φ given by (29), the topological charge, which is the volume integral of ρ in (28), is

$$q = f_\infty^2[\eta^2 - (k_\infty^2 + \tfrac{3}{5}f_\infty^2)] \tag{30}$$

up to normalization. Since $f_\infty^2 + k_\infty^2 = \eta^2$ from (2), (30) can be rewritten as

$$q = \tfrac{2}{5}f_\infty^5 \tag{30′}$$

which is nontrivial provided that $f_\infty \neq 0$.

Therefore solutions with topological charge q and finite action must exist in this case. On the other hand these solutions are non-minimal, because it was readily verified that substituting the ansatz (29) into the Bogomoln'yi equation (27) yields $f(r)=k(r)=0$.

We now return to the *minimal models*. These are guaranteed not to be overdetermined by the fact that their Bogomolniy equations, given below, are of zero tensor rank,

$$\sqrt{V(S)} = \varepsilon_{i_1 \ldots i_d} \Phi_{i_1 \ldots i_d} \tag{31}$$

which clearly saturate the topological inequality,

$$\int [\Phi^2_{i_1 \ldots i_d} + V(S)]\, \mathrm{d}^d x \geqslant \frac{2}{d!} \int \varepsilon_{i_1 \ldots i_d} \Phi_{i_1 \ldots i_d} \sqrt{V}\, \mathrm{d}^d x. \tag{32}$$

The Lagrange density of these models, defined on $\mathbb{R}_d$, is

$$\mathscr{L}_d = \mathrm{tr}\, \Phi^2_{i_1 \ldots i_d} + V. \tag{33}$$

Equations (31) defined on $\mathbb{R}_d$, can be integrated if we choose the matrix valued fields Φ to be

$$\Phi = \mathrm{i} \Phi^a \Gamma_a \tag{34}$$

where $a = 1, \ldots, d$, and Γ_a are the $2^{d/2} \times 2^{d/2}$ Γ-matrices in d dimensions. Here the case $d = 3$ is privileged, because $\Gamma_a = \sigma_a$ actually belong to the algebra of SO(3) and SU(2). As such, $\mathscr{L}_3$ given by (33) is potentially a very useful model for physical applications.

Substituting (34) into (31), and using the fact that the antisymmetrized d-fold product of d-dimensional Γ-matrices is equal to $\varepsilon^{1_1 \cdots a_d}$ times the unit matrix for d odd, and to $\varepsilon^{a_1 \cdots a_d}$ times the chirality matrix Γ_{d+1} for d even, (31) reduces to

$$\sqrt{V} = \det \left\| \frac{\partial \Phi}{\partial x_i} \right\| \cdot \mathbb{1} \qquad \text{odd } d \tag{35a}$$

$$= \det \left\| \frac{\partial \Phi}{\partial x_i} \right\| \cdot \Gamma_{d+1} \qquad \text{even } d \tag{35b}$$

where $\|\partial \Phi_i / \partial x_j\|$ is the Jacobian of transformation between x_i and Φ_i. For example, in the physically most important example in $d = 3$ with $\Phi \in \mathrm{su}(2)$, the self-duality equation (31) is

$$\sqrt{V} = \varepsilon_{ijk} \partial_i \phi^a \partial_j \phi^b \partial_k \phi^c \varepsilon_{abc} \tag{36}$$

according to (34).

Then choosing $\sqrt{V}$ to have the appropriate matrix structure $\mathbb{1}$ in every odd-dimension equation (35a) reduces to a number valued equation.

Remembering then that $V = V(\eta, \Phi^a \Phi^a)$ in the case (34) at hand, we see that $V = V(\eta, \phi^2)$ is a function only of the magnitude ϕ of the d-component field Φ^a. It is therefore natural to decompose Φ^a into its polar parametrization. Denoting the magnitude of Φ^a by ϕ, the polar parameters $\Theta_1, \ldots, \Theta_{d-2}$, and the azimuthal parameter by Ψ, equation (35) can be integrated formally.

Next we can choose to identify $\Theta_a = \theta_a$, the $(d-2)$ polar angles of $\mathbb{R}_d$, and, $\Psi = n\phi$ with $n =$ integer and ϕ the azimuthal angle of $\mathbb{R}_d$. This amounts to a spherically symmetric 'hedgehog' field configuration with $\phi = \phi(r)$. In that case the integral of (35) reduces to

$$n \int \frac{\phi^{d-1}\, \mathrm{d}\phi}{\sqrt{V(\phi^2, \eta)}} = \left(\frac{r^d}{d} \right) + \text{constant} \tag{37}$$

which can be evaluated once $V(\eta^2, \phi^2)$ is specified. The result is an explicit minimal-action solution with winding number n, provided that we choose the constant of integration in (37) to be zero. With this choice of constant, and with a symmetry breaking potential $V(\eta, \phi)$, it can easily be verified that

$$\phi(r) \xrightarrow[r\to 0]{} 0. \tag{38}$$

The behaviour (38) is a necessary condition for the topological charge given by the right-hand side of (32) to exist. For example in the $d=3$ case, for the current Ω_i, defined by the topological density, to be well defined [18].

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